

AD-A144 361

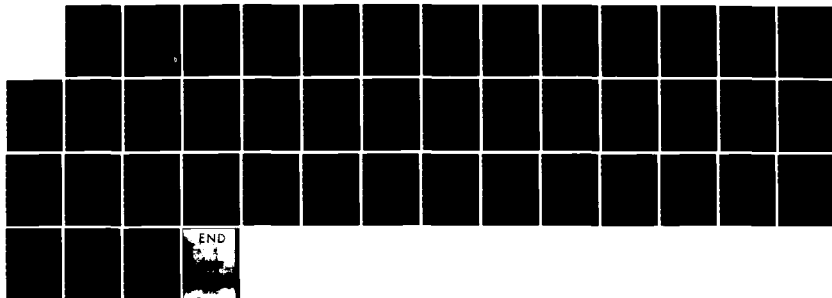
SPATIO-TEMPORAL SPECTRAL ANALYSIS BY EIGENSTRUCTURE  
METHODS(U) STANFORD UNIV CA INFORMATION SYSTEMS LAB  
M WAX ET AL. 1984 AFOSR-TR-84-0487 AFOSR-83-0228

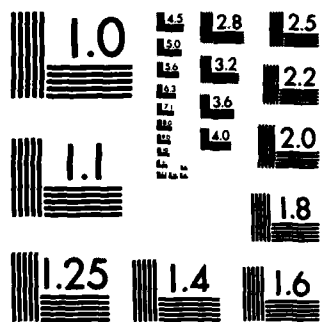
1/1

UNCLASSIFIED

F/G 12/1

NL





MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

UNCLASSIFIED

AD-A144 361

(12)

SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b. RESTRICTIVE MARKINGS													
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.													
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE															
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) <b>AFOSR-TR- 84 - 0487</b>													
6a. NAME OF PERFORMING ORGANIZATION Stanford University	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research													
6c. ADDRESS (City, State and ZIP Code) Department of Electrical Engineering Stanford CA 94305		7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332													
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR	8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-83-0228													
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332		10. SOURCE OF FUNDING NOS. <table border="1"><tr><td>PROGRAM ELEMENT NO. 61102F</td><td>PROJECT NO. 2304</td><td>TASK NO. A6</td><td>WORK UNIT NO.</td></tr></table>		PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2304	TASK NO. A6	WORK UNIT NO.								
PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2304	TASK NO. A6	WORK UNIT NO.												
11. TITLE (Include Security Classification) SPATIO-TEMPORAL SPECTRAL ANALYSIS BY EIGENSTRUCTURE METHODS.															
12. PERSONAL AUTHOR(S) M. Wax, T.J. Shan and T. Kailath															
13a. TYPE OF REPORT Technical	13b. TIME COVERED FROM TO	14. DATE OF REPORT (Yr., Mo., Day) 1984	15. PAGE COUNT 42												
16. SUPPLEMENTARY NOTATION															
17. COSATI CODES <table border="1"><tr><td>FIELD</td><td>GROUP</td><td>SUB. GR.</td></tr><tr><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td></tr></table>		FIELD	GROUP	SUB. GR.										18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)  T <sub>h</sub> 52	
FIELD	GROUP	SUB. GR.													
19. ABSTRACT (Continue on reverse if necessary and identify by block number) <p>This paper presents new algorithms for estimating the spatio-temporal spectrum of the signals received by a passive array. The algorithms are based on the eigenstructure of the covariance and spectral-density matrices of the received signals. They allow partial correlation between the sources and thus are applicable to certain kinds of multipath problems. Simulation results that illustrate the performance of the new algorithms are presented.</p>															
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>		21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED													
22a. NAME OF RESPONSIBLE INDIVIDUAL CPT Brian W. Woodruff		22b. TELEPHONE NUMBER (Include Area Code) (202) 767- 5027	22c. OFFICE SYMBOL NM												

DTIC FILE COPY

84 06 28 12 E

DTIC  
ELECTE  
AUG 17 1984

AFOSR-TR- 84-0487

**SPATIO-TEMPORAL SPECTRAL ANALYSIS  
BY EIGENSTRUCTURE METHODS<sup>†</sup>**

Mati Wax, Tie-Jun Shan and Thomas Kailath  
Information Systems Laboratory  
Stanford University  
Stanford, CA 94305

**ABSTRACT**

This paper presents new algorithms for estimating the spatio-temporal spectrum of the signals received by a passive array. The algorithms are based on the eigenstructure of the covariance and spectral-density matrices of the received signals. They allow partial correlation between the sources and thus are applicable to certain kinds of multipath problems. Simulation results that illustrate the performance of the new algorithms are presented.



Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

**DTIC  
ELECTE  
AUG 17 1984  
S D E**

Approved for public release;  
distribution unlimited.

Grant AFOSR-83-0228

<sup>†</sup>This work was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command under Contract AF49-820-79-C-0059, the U.S. Army Research Office, under Contract DAAG29-79-C-0215, and by the Joint Services Program at Stanford University under Contract DAAG29-81-K-0057.

84 06 28 012

## I. Introduction

In radar, sonar and seismology one is frequently interested in estimating the directions-of-arrival and the spectral densities of radiating sources from measurements provided by a passive array of sensors. In the simplest case, the signals received by the sensors consist of scaled and delayed replicas of the waveform radiated by a single source. In a more complicated scenario, there may be multiple sources and multiple propagation paths from the sources to the sensors.

The problem of the simultaneous estimation of the directions-of-arrival and the spectral densities of the impinging sources can be regarded as a 2-D (two-dimensional) spectral estimation problem. Given spatial and temporal samples of the received signals, the problem is to estimate the 2-D spectrum, or the energy distribution, in both the spatial (wavenumber) and temporal (frequency) domains, i.e, in the wavenumber-frequency plane. The spatial spectrum consists of point masses at the different wavenumbers. The temporal spectrum may consist of point masses at different frequencies, in the case of narrowband sources, or may be continuous in the case of wideband sources. When the spectral-densities of the sources are known, the problem of direction-of-arrival estimation reduces to a 1-D spectral estimation problem in the spatial domain.

Since the number of samples (i.e. sensors) in the spatial domain is usually small, classical Fourier analysis yields low spatial resolution. As a result, alternative methods that provide higher resolution have been developed. The Maximum Entropy (ME) method of Burg (1967) and the Minimum Variance (MV) method of Capon (1969) were the first methods that provided increased resolution in the

spatial domain. The Minimum Variance method was originally developed for the 2-D problem, while the Maximum Entropy method was originally developed for the 1-D problem. Different extensions of the Maximum Entropy method to the 2-D problem have been proposed by Roucos and Childers (1980), Lim and Malik (1981) and McClellan (1982) among others. A different high resolution method for the 2-D problem, known as 2-D linear-prediction, was developed by Jackson and Chien (1978), Jain and Ranganath (1978), Frost and Sullivan (1979) and Kumareseran and Tufts (1981). Recently, another method for the 2-D problem based on ARMA modeling of the received signals has been proposed by Morf et al (1979), and further developed and elaborated by Porat and Friedlander (1983) and Nehorai and Morf (1983). We should note that the Maximum-Likelihood estimator, though asymptotically optimal, has not been advocated for the 2-D problem because of the high computational complexity involved. In the somewhat simpler 1-D problem, where the spectral densities of the sources are known, the Maximum-Likelihood estimator has been studied by Good (1963), Hahn and Tretter (1973), Schweppe (1968) and Wax and Kailath (1983a).

In the special case that the sources are narrowband and have the same known center-frequency, the problem of direction-of-arrival estimation reduces to the problem of 1-D *harmonic-retrieval*. A method, tailored especially for this problem, that provides better resolution than offered by the Minimum Variance, Maximum Entropy and the linear-prediction methods, was pioneered by Pisarenko (1973). The method is based on the eigenstructure of the covariance matrix of the received signals. Schmidt (1979) (1981), and independently Bienvenu and Kopp (1980, 1981), have improved the resolution of Pisarenko's

method, and extended it from the restricted uniform linear array case to a general array. Schmidt's method - as Pisarenko's - is a time-domain method based on the eigenstructure of the covariance matrix of the received signals; Bienvenu and Kopp's method is a frequency-domain method based on the eigenstructure of the spectral density matrix of the received signals. Related but somewhat different eigenstructure methods for the 1-D problem have been proposed by Ligget (1972), Owsley (1973), Reddi (1979), Johnson and Degraff (1982), Kumareseran and Tufts (1983) and Bronez and Cadzow (1983). The methods described above are off-line; on-line implementations of Pisarenko's method have been presented by Thompson (1979), Cantoni and Godara (1980) and Reddy et al. (1982a). Extension of these on-line methods to the 2-D harmonic retrieval problem have been presented by Larimore (1981) and Reddy et. al (1982b).

In this paper we extend the eigenstructure approach to the 2-D problem. We present a time-domain eigenstructure method for the 2-D harmonic retrieval problem, that is for the simultaneous estimation of both the direction-of-arrival and the center-frequency of narrowband sources, and a frequency-domain eigenstructure method for the estimation of the direction-of-arrival and the spectral density matrix of wideband sources.

The narrowband and the wideband 2-D problems are formulated in section II. The time-domain eigenstructure method for the narrowband problem is presented in section III. The frequency-domain eigenstructure method for the wideband problem is presented in section IV. Simulation results that illustrate the performance of the proposed algorithms are presented in section V.

## II. Problem Formulation

The formulation will be somewhat different for the narrowband and wideband problems. Following the convention in array processing, a problem will be referred to as a narrowband problem if the bandwidth of the impinging sources is much smaller than the reciprocal of the propagation time of the signal across the array; otherwise it will be referred to as wideband.

### A. The Narrowband Problem

Consider an array with  $m$  sensors, each followed by a tapped-delay-line with  $p$  taps spaced  $D$  delay units apart. Assume that  $d$  narrowband sources ( $d < mp$ ) centered at frequencies  $\omega_1, \dots, \omega_d$ , impinge on the array from directions  $\theta_1, \dots, \theta_d$  (see Fig. 1). Assume that the signals emitted by the sources are stationary zero-mean narrowband stochastic processes. The signals received by the sensors are scaled and delayed replicas of the radiated signals. Then, using complex (analytic) signal representation, the output of the  $h$ -th delay unit of the  $i$ -th sensor, can be expressed as

$$r_i(t - hD) = \sum_{k=1}^d a_{ik} s_k(t - \tau_{ik} - hD) + n_i(t - hD) \quad (1)$$

where  $s_k(\cdot)$  is the signal radiated by the  $k$ -th source as observed at an arbitrarily chosen reference point,  $\omega_k$  is the center frequency of the  $k$ -th source,  $\tau_{ik}$  is the propagation delay between the  $i$ -th sensor and the reference point for the  $k$ -th source,  $a_{ik}$  is the amplitude response of the  $i$ -th sensor to the  $k$ -th source, and  $n_i(\cdot)$  is the additive noise at the  $i$ -th sensor.



Since  $s_k(\cdot)$  is a narrowband process we can well approximate the time-delay by a phase-shift. Therefore, we can rewrite (1) as

$$r_i(t - hD) = \sum_{k=1}^d a_{ik} s_k(t) e^{-j\omega_i r_{s_k} - j\omega_i hD} + n_i(t - hD) \quad \begin{matrix} i=1,\dots,m \\ h=0,\dots,p-1 \end{matrix} \quad (2)$$

Assume that the received signals are sampled simultaneously at times  $t_k$  ( $k = 1, \dots, N$ ), yielding  $N$  "snapshots", each consisting of  $mp$  samples  $r_i(t_k - hD)$  ( $i = 1, \dots, m$ ;  $h = 0, \dots, p-1$ ). Grouping the samples corresponding to the  $i$ -th sensor into a  $p \times 1$  vector, we can rewrite equation (2) in matrix form

$$\mathbf{r}_i(t) = \mathbf{A}_i \mathbf{s}(t) + \mathbf{n}_i(t) \quad i = 1, \dots, m \quad (3.a)$$

where  $\mathbf{r}_i(t)$  and  $\mathbf{n}_i(t)$  are the  $p \times 1$  vectors

$$\mathbf{r}_i^T(t) = [ r_i(t) \ \cdots \ r_i(t - (p-1)D) ] \quad (3.b)$$

$$\mathbf{n}_i^T(t) = [ n_i(t) \ \cdots \ n_i(t - (p-1)D) ] \quad (3.c)$$

$\mathbf{s}(t)$  is the  $d \times 1$  vector

$$\mathbf{s}^T(t) = [ s_1(t) \ \cdots \ s_d(t) ] \quad (3.d)$$

and  $\mathbf{A}_i$  is the  $p \times d$  matrix

$$\mathbf{A}_i = \begin{bmatrix} a_{i1} e^{-j\omega_i r_{s1}} & \cdots & a_{id} e^{-j\omega_i r_{sd}} \\ \vdots & & \vdots \\ a_{i1} e^{-j\omega_i r_{s1} - j\omega_i (p-1)D} & \cdots & a_{id} e^{-j\omega_i r_{sd} - j\omega_i (p-1)D} \end{bmatrix} \quad (3.e)$$

Stacking the  $p \times 1$  vectors  $\mathbf{r}_i(t)$  ( $i = 1, \dots, m$ ) into a  $mp \times 1$

"snapshot" vector  $\mathbf{r}(t)$ , we can further simplify the notation and rewrite equation (3) as

$$\mathbf{r}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \quad (4.a)$$

where  $\mathbf{r}(t)$  and  $\mathbf{n}(t)$  are the  $mp \times 1$  vectors

$$\mathbf{r}^T(t) = [\mathbf{r}_1^T(t) \cdots \mathbf{r}_m^T(t)] \quad (4.b)$$

$$\mathbf{n}^T(t) = [\mathbf{n}_1^T(t) \cdots \mathbf{n}_m^T(t)] \quad (4.c)$$

and  $\mathbf{A}$  is the  $mp \times d$  matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 \\ \vdots \\ \mathbf{A}_m \end{bmatrix} = \begin{bmatrix} a_{11}e^{-j\omega_1 r_{11}} & \cdots & a_{1d}e^{-j\omega_d r_{1d}} \\ \vdots & & \vdots \\ a_{m1}e^{-j\omega_1 r_{m1} - j\omega_1(p-1)D} & \cdots & a_{md}e^{-j\omega_d r_{md} - j\omega_d(p-1)D} \end{bmatrix} \quad (4.d)$$

Note that each column of  $\mathbf{A}$  is associated with a different source. We shall denote these column vectors by  $\mathbf{A}_{\theta, \omega_k}$  ( $k = 1, \dots, d$ ), and refer to them as the *direction-frequency vectors* of the sources.

Multiplying (4.a) by its conjugate transpose and taking expectations, assuming that the noises are zero-mean with variance  $\sigma^2$ , that their correlation time is smaller than  $D$ , and that they are uncorrelated with each other and with the source signals, we obtain

$$\mathbf{R} = \mathbf{A}\mathbf{S}\mathbf{A}^\dagger + \sigma^2\mathbf{I} \quad (5.a)$$

where  $\dagger$  denotes the conjugate transpose,  $\mathbf{R}$  is the covariance matrix of the received signals,  $\mathbf{I}$  is the identity matrix, and  $\mathbf{S}$  is the covariance matrix of the sources, i.e.,

$$\mathbf{S} = E[ \mathbf{s}(t) \mathbf{s}^{\dagger}(t) ] . \quad (5.b)$$

where  $E$  denotes the expectation operator.

We shall assume that  $\mathbf{S}$  is *nonsingular*. That is, no source is a scaled and delayed version of any other source; in other words, no two sources are *fully correlated*. We do, however, allow for sources to be *partially* correlated, in the sense that the covariance matrix of the sources may be nondiagonal, as long as it is nonsingular. This allowance makes the model applicable to certain kinds of *multipath* problems, e.g. those where the reflection introduces some random perturbation to the multipath signal.

## B. The Wideband Problem

Consider, as before, an array with  $m$  sensors. Assume now that  $d$  *wideband* sources ( $d < m$ ), with *identical* bandwidth  $B$ , impinge on the array from directions  $\theta_1, \dots, \theta_d$ . The signals emitted by the sources are assumed to be stationary zero-mean stochastic processes. Using the notation of (1), the signal received at the  $i$ -th sensor can be expressed as

$$r_i(t) = \sum_{k=1}^d a_{ik} s_k(t - \tau_{ik}) + n_i(t) \quad (6)$$

Unlike the narrowband problem, where we have formulated the problem in terms of the sampled data, it turns out that for the wideband problem it is more convenient to formulate the problem in terms of the continuous signals. Assuming that we are observing the received signals over a finite interval  $T$ , it follows that we can approximately represent the received signal  $r_i(\cdot)$  by a Fourier-series

$$r_i(t) = \sum_{n=l}^{l+M} R_i(\omega_n) e^{j\omega_n t} \quad (7.a)$$

where  $R_i(\omega_n)$  are the Fourier-coefficients, given by

$$R_i(\omega_n) = \frac{1}{T^{1/2}} \int_{-T/2}^{T/2} r_i(t) e^{j\omega_n t} dt \quad (7.b)$$

and

$$\omega_n = \frac{2\pi}{T} n \quad n = l, \dots, l+M \quad (7.c)$$

where  $\omega_l$  and  $\omega_{l+M}$  are the lowest and highest frequencies, respectively, included in the bandwidth  $B$ . Note that there are only positive frequencies since we are using the complex (analytic) signal representation, and that there are a finite number of them since we are assuming that the signals have an approximately defined finite bandwidth  $B$ .

Then representing both sides of (6) by their Fourier-coefficients, assuming that the observation time is much longer than the propagation time of the signals across the array ( $\tau_{ik} \ll T$ ) so that to a good approximation the time-delay transforms to phase-shift in the Fourier-domain, we obtain

$$R_i(\omega_n) = \sum_{k=1}^d a_{ik} e^{-j\omega_n \tau_{ik}} S_k(\omega_n) + N_i(\omega_n) \quad (8)$$

In matrix notation this becomes

$$\mathbf{R}(\omega_n) = \mathbf{A}(\omega_n) \mathbf{S}(\omega_n) + \mathbf{N}(\omega_n) \quad (9.a)$$

where  $\mathbf{R}(\omega_n)$  and  $\mathbf{N}(\omega_n)$  are the  $m \times 1$  vectors

$$\mathbf{R}^T(\omega_n) = [R_1(\omega_n) \cdots R_m(\omega_n)] \quad (9.b)$$

$$\mathbf{N}^T(\omega_n) = [N_1(\omega_n) \cdots N_m(\omega_n)] \quad (9.c)$$

$\mathbf{S}(\omega_n)$  is the  $d \times 1$  vector

$$\mathbf{S}^T(\omega_n) = [S_1(\omega_n) \cdots S_d(\omega_n)] \quad (9.d)$$

and  $\mathbf{A}(\omega_n)$  is the  $m \times d$  matrix

$$\mathbf{A}(\omega_n) = \begin{bmatrix} a_{11}e^{-j\omega_n\tau_{11}} & \cdots & a_{1d}e^{-j\omega_n\tau_{1d}} \\ \vdots & & \vdots \\ a_{m1}e^{-j\omega_n\tau_{m1}} & \cdots & a_{md}e^{-j\omega_n\tau_{md}} \end{bmatrix} \quad (9.e)$$

Note that each column of  $\mathbf{A}(\omega_n)$  is associated with a different source. We shall denote these column vectors by  $\mathbf{A}_k(\omega_n)$  ( $k = 1, \dots, d$ ), and refer to them as the *direction vectors* of the sources.

Multiplying (9.a) by its conjugate transpose, and taking expectations, assuming that the noises are zero-mean and uncorrelated with the signals, we obtain

$$E[\mathbf{R}(\omega_n)\mathbf{R}^\dagger(\omega_n)] = \mathbf{A}(\omega_n)E[\mathbf{S}(\omega_n)\mathbf{S}^\dagger(\omega_n)]\mathbf{A}^\dagger(\omega_n) + E[\mathbf{N}(\omega_n)\mathbf{N}^\dagger(\omega_n)] \quad (10)$$

Next, assuming that the observation time is large compared to the correlation time of the processes involved, the covariance matrix of the Fourier-coefficient vector will be approximately equal to the spectral density matrix (see, e.g., Whalen (1971) p.81). Thus, assuming that the noises are uncorrelated with each other and have the same spectral densities, we can rewrite (10) as

$$\mathbf{K}(\omega_n) = \mathbf{A}(\omega_n) \mathbf{P}(\omega_n) \mathbf{A}^\dagger(\omega_n) + \sigma^2(\omega_n) \mathbf{I} \quad (11)$$

where  $\mathbf{K}(\omega_n)$  and  $\mathbf{P}(\omega_n)$  are the spectral density matrices of the processes  $\{r_i(\cdot)\}_{i=1}^m$  and  $\{s_i(\cdot)\}_{i=1}^d$ , respectively, and  $\sigma^2(\omega_n)$  is the spectral density of the noises  $\{n_i(\cdot)\}_{i=1}^m$ .

We shall assume that  $\mathbf{P}(\omega_n)$  is *nonsingular*, i.e., no source is a scaled and delayed version of any other source. In the narrowband case, we allow for sources to be *partially* correlated, in the sense that the spectral-density matrix of the sources may be nondiagonal.

### III. Time-domain Eigenstructure Method for the Narrowband Problem

The solution we are going to present to the narrowband problem is based on the the eigenstructure of the covariance matrix  $\mathbf{R}$  and therefore is referred to as a time-domain method. It presumes that the following conditions hold:

- (I) The covariance matrix of the sources  $\mathbf{S}$  is *nonsingular*.
- (II) Any set of  $d+1$  direction-frequency vectors are *linearly independent*.

With these assumptions it can be easily verified that the eigenvalues and eigenvectors of  $\mathbf{R}$ , denoted by  $\{\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{mp}\}$  and  $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{mp}\}$ , respectively, have the following properties:

The minimal eigenvalue of  $\mathbf{R}$  is  $\sigma^2$  with *multiplicity*  $mp - d$ , i.e. ,

$$\lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_{mp} = \sigma^2 \quad (12)$$

The eigenvectors corresponding to the minimal eigenvalue are *orthogonal* to the columns of the matrix  $\mathbf{A}$ , i.e. ,

$$\mathbf{A}^T \mathbf{V}_i = 0 \quad i = d+1, \dots, mp \quad (13.a)$$

or, more explicitly

$$\{\mathbf{V}_{d+1}, \dots, \mathbf{V}_{mp}\} \perp \{\mathbf{A}_{\theta, \omega_1}, \dots, \mathbf{A}_{\theta, \omega_d}\} \quad (13.b)$$

where  $\mathbf{A}_{\theta, \omega_i}$  is the  $i$ -th column of  $\mathbf{A}$ , referred to as the *direction-frequency vector* of the  $i$ -th source.

Note that for properties (12) and (13) to hold, it is only required that  $d$ , rather than  $d+1$  as in condition (II) above, direction-frequency vectors be

*linearly independent*. However, the stronger condition (II) is needed to assure the uniqueness of the solution obtained by the eigenstructure method, as we shall presently show.

The eigenstructure method is based on straightforward exploitation of properties (12) and (13). Observe, first, that it follows from (12) that the *number* of sources can be determined from the *multiplicity* of the smallest eigenvalue. Second, it follows from the *orthogonality* relation (13) between the direction-frequency vectors of the impinging sources and the *eigenvectors* corresponding to the *minimal eigenvalue*, that the *directions-of-arrival* and the *center-frequencies* of the sources can be determined simply by *searching* for those direction-frequency vectors that are orthogonal to the eigenvectors corresponding to the minimal eigenvalue. Note that to avoid ambiguities, no direction-frequency vector other than those corresponding to the impinging sources, should be orthogonal to the eigenvectors corresponding to the minimal eigenvalue. That will be the case if the span of the  $d$  direction-frequency vectors of the impinging sources does not include any other direction-frequency vector. This implies that to assure unambiguous results, *any* set of  $d+1$  direction-frequency vectors should be *linearly independent*, as stated in condition (II) above.

#### A. Determination of the Number of Sources

The method for determining the number of sources we have outlined above was based on the eigenvalues of the *true* covariance matrix. In practice, however, the true covariance matrix is *unknown*. To apply the method we must, therefore,



*estimate* the eigenvalues from the data. The problem is that the estimated eigenvalues will *not* obey relation (12). With probability one, the small  $mp-d$  eigenvalues will all be *different*, somehow clustered around their true value. Thus it is impossible to determine the number of sources simply by observing the small eigenvalues, and a more sophisticated procedure, based on statistical considerations is needed.

Such a procedure was developed by Bartlett and Lawley (Bartlett (1954), Lawley (1956)) in the context of factor analysis. The Bartlett-Lawley procedure takes the form of a sequence of nested *hypothesis tests* for the multiplicity of the smallest eigenvalue, starting from the highest possible multiplicity and testing successively down to the lowest possible multiplicity. At the  $k$ 'th step of the procedure ( $k = 0, \dots, mp-1$ ), the null hypothesis that the smallest eigenvalue of the covariance matrix has multiplicity  $mp - k$ , namely,

$$H_k : \lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_{mp}$$

is tested. The likelihood-ratio for this problem, under Gaussian assumptions, is given by

$$Q_k = \left( \frac{\prod_{i=k+1}^{mp} l_i}{\left( \frac{1}{mp-k} \sum_{i=k+1}^{mp} l_i \right)^{mp-k}} \right)^N \quad (14)$$

where  $l_1 > l_2 > \dots > l_{mp}$  are the eigenvalues of the sample-covariance matrix  $\hat{R}$ , defined by

$$\hat{R} = \frac{1}{N} \sum_{i=1}^N \mathbf{r}(t_i) \mathbf{r}(t_i)^T \quad (15)$$

Note that  $Q_k$  is a monotonic function of the ratio of the arithmetic-mean and the geometric-mean of the smallest  $m - k$  eigenvalues whose equality is tested.

The exact distribution of the statistic  $Q_k$  under the null hypothesis  $H_k$  is unknown. However asymptotically, as  $N \rightarrow \infty$ , it follows from the general theory of likelihood-ratios (see e.g Cox and Hinckly (1974)) that the statistic  $-2\ln Q_k$  has a  $\chi^2$  distribution with  $(mp-k)^2 - 1$  degrees of freedom. Therefore, if  $N$  is large an approximate test of size  $\alpha$  of  $H_k$  is to reject  $H_k$  if  $-2\ln Q_k > c(\alpha; (mp-k)^2 - 1)$ , where  $c(\alpha; r)$  is the upper  $100\alpha\%$  point of the  $\chi_r^2$  distribution. Note that the size  $\alpha$  of the test is a parameter left to the subjective decision of the designer.

The hypotheses  $H_k$  ( $k = 0, 1, \dots, mp-1$ ) are tested sequentially. The value of  $k$  for which  $H_k$  is first accepted is selected as the estimate  $\hat{d}$  of the number of sources. For the 1-D problem ( $p=1$ ) this procedure reduces to that presented by Simkins (1980).

We should note that a different approach to the problem, based on applying the information theoretic criteria for statistical model identification introduced by Akaike, Schwartz and Rissanen, was recently presented by Wax and Kailath (1983b). In this approach no subjective judgement is required in the decision process.

## B. Estimation of the Eigenstructure

The method we have outlined for the determination of the directions-of-arrival and the center-frequencies was based on the eigenstructure of the true covariance matrix. The problem is that in practice the true covariance matrix and hence its eigenstructure are *unknown*. To apply the method we must, therefore, *estimate* the eigenstructure from the data. To this end, let the eigenvalues and eigenvectors of the sample-covariance matrix  $\hat{\mathbf{R}}$ , defined in (15), be given by  $l_1 > l_2 > \dots > l_{mp}$  and  $\mathbf{C}_1, \dots, \mathbf{C}_{mp}$ , respectively. Now, using the estimate of the number of sources  $\hat{d}$ , it follows from Anderson (1963) that the maximum likelihood estimates of the eigenvalues and eigenvectors of  $\mathbf{R}$  are given by

$$\hat{\lambda}_i = l_i \quad i = 1, \dots, \hat{d} \quad (16.a)$$

$$\hat{\sigma}^2 = \frac{1}{mp - \hat{d}} \sum_{i=\hat{d}+1}^{mp} l_i \quad (16.b)$$

$$\hat{\mathbf{V}}_i = \mathbf{C}_i \quad i = 1, \dots, \hat{d} \quad (16.c)$$

and up to an orthogonal transformation from the right (the eigenvectors corresponding to multiple eigenvalue define a subspace and are nonunique up to an orthogonal transformation)

$$\hat{\mathbf{V}}_i = \mathbf{C}_i \quad i = \hat{d} + 1, \dots, mp \quad (16.d)$$

### C. Estimation of the Directions-Of-Arrival and Center-Frequencies

With estimates of the number of sources and the eigenvectors we now proceed to the simultaneous estimation of the directions-of-arrival and the center-frequencies of the sources. We have seen that the direction-frequency vectors corresponding to the  $d$  sources are orthogonal to the true eigenvectors  $\mathbf{V}_{d+1}, \dots, \mathbf{V}_{mp}$  corresponding to the repeated smallest eigenvalue. However, since we only have *estimates* of these eigenvectors, the orthogonality relation does *not* hold any more. Instead, the cosine of the angle between each of the direction-frequency vectors  $\mathbf{A}_{\theta_1 \omega_1}, \dots, \mathbf{A}_{\theta_d \omega_d}$  and each of the eigenvectors  $\hat{\mathbf{V}}_{d+1}, \dots, \hat{\mathbf{V}}_{mp}$  will probably be "close" to zero, that is

$$\cos \alpha_{ik}^2 = |\mathbf{A}_{\theta_i \omega_i} \hat{\mathbf{V}}_i|^2 / |\mathbf{A}_{\theta_i \omega_i} \mathbf{A}_{\theta_i \omega_i}| \approx 0, \quad \begin{matrix} k = 1, \dots, d \\ i = d+1, \dots, mp \end{matrix} \quad (17)$$

where  $|\cdot|$  denotes the modulus of the complex number. Thus, if  $\mathbf{A}_{\theta \omega}$  denotes the direction-frequency vector corresponding to a source at direction-of-arrival  $\theta$  and center-frequency  $\omega$ , then the  $d$  sources should be chosen as those whose direction-frequency vectors are "most nearly orthogonal" to the set of eigenvectors  $\{\hat{\mathbf{V}}_{d+1}, \dots, \hat{\mathbf{V}}_{mp}\}$  corresponding to the minimal eigenvalue.

There are many metrics that can be applied to measure this "distance from orthogonality". One metric is the arithmetic-mean of the square cosines of the angle between the direction-frequency vector and the eigenvectors corresponding to the smallest eigenvalue, that is,  $\frac{1}{mp-d} \sum_{i=d+1}^{mp} |\mathbf{A}_{\theta \omega} \hat{\mathbf{V}}_i|^2 / |\mathbf{A}_{\theta \omega} \mathbf{A}_{\theta \omega}|$ .

Another metric is the geometric-mean of these quantities, that is

$\prod_{i=d+1}^{mp} |\mathbf{A}^T \omega \hat{\mathbf{V}}_i|^{\frac{2}{mp-d}} / |\mathbf{A}^T \omega \mathbf{A}_{\theta, \omega}|$ . Both these metrics are two special cases of the general family (see e.g. Pisarenko (1972)) given by  $(\frac{1}{mp-d} \sum_{i=d+1}^{mp} |\mathbf{A}^T \omega \hat{\mathbf{V}}_i|^{2r})^{\frac{1}{r}} / |\mathbf{A}^T \omega \mathbf{A}_{\theta, \omega}|$ . Clearly, the arithmetic-mean corresponds to  $r = 1$ . The geometric-mean can be shown to correspond to the limit obtained as  $r \rightarrow 0$ .

With these "orthogonality metrics", we can now proceed to the estimation of the direction-of-arrival and the center-frequencies of the sources. Note that since we are interested only in the extremal points of these metrics, we can extract the information from any monotonic function of them. In order to resemble the form of Capon's Minimum Variance (MV) and Burg's Maximum Entropy (ME) estimators, it seems natural to choose the inverse of these metrics as the representative form. For the arithmetic-mean metric, we therefore plot

$$J_1(\theta, \omega) = \frac{|\mathbf{A}^T \omega \mathbf{A}_{\theta, \omega}|}{\frac{1}{mp-d} \sum_{i=d+1}^{mp} |\mathbf{A}^T \omega \hat{\mathbf{V}}_i|^2} \quad (18.a)$$

For the geometric-mean metric, we plot

$$J_0(\theta, \omega) = \frac{|\mathbf{A}^T \omega \mathbf{A}_{\theta, \omega}|}{\prod_{i=d+1}^{mp} |\mathbf{A}^T \omega \hat{\mathbf{V}}_i|^{\frac{2}{mp-d}}} \quad (18.b)$$

and for the general-mean metric, we plot

$$J_r(\theta, \omega) = \frac{|A_{\theta, \omega}|}{\left( \frac{1}{mp-d} \sum_{i=d+1}^{mp} |A_{\theta, \omega} \hat{V}_i|^{2r} \right)^{\frac{1}{r}}} \quad (18.c)$$

The directions-of-arrival and the center-frequencies of the impinging sources are determined as the  $\hat{d}$  points  $(\hat{\theta}_1, \hat{\omega}_1), \dots, (\hat{\theta}_d, \hat{\omega}_d)$  that yield the highest peaks of either of these 2-D functions.

The estimators presented above differ in their resolution and accuracy properties. The arithmetic-mean estimator will have a peak in the point  $(\theta, \omega)$  if and only if  $A_{\theta, \omega}$  is "almost orthogonal" to *all* the eigenvectors  $\hat{V}_{d+1}, \dots, \hat{V}_{mp}$ . The geometric-mean estimator, on the other hand, will have a significant peak at the point  $(\theta, \omega)$  even if  $A_{\theta, \omega}$  is "almost orthogonal" to only *one* of the eigenvectors  $\hat{V}_{d+1}, \dots, \hat{V}_{mp}$ . Therefore, the geometric-mean estimator will have higher resolution but lower accuracy than the arithmetic-mean estimator.

More delicate trade-offs between resolution and accuracy can be obtained by using the general estimator (18.c) with the parameter  $r$  chosen appropriately.

We should note that Sharman et. al (1983) have recently shown that the arithmetic-mean metric (18.a) is asymptotically optimal for the case of a single source. That is, under a small error assumption, the bias of the estimator (18.a) approaches zero, and its variance approaches the Cramer-Rao lower bound.

#### D. Estimation of the Covariance Matrix of the Sources

The source covariance matrix  $S$  contains valuable information on the

impinging sources. The diagonal elements indicate the power of the sources, and the off-diagonal elements indicate the cross-correlation between them. This information can be valuable for distinguishing between a direct path and multipath, for example.

Using the estimates of the number, the directions-of-arrival and the center-frequencies of the sources, the estimate of the covariance matrix of the sources follows immediately from the works of Schmidt (1979) and of Bienvenu and Kopp (1981). Note first that (12) implies that

$$\mathbf{A} \mathbf{S} \mathbf{A}^{\dagger} = \sum_{i=1}^d (\lambda_i - \sigma^2) \mathbf{V}_i \mathbf{V}_i^{\dagger} \quad (19)$$

Solving this equation for  $\mathbf{S}$ , and substituting estimated quantities for the true quantities, yields

$$\hat{\mathbf{S}} = (\hat{\mathbf{A}}^{\dagger} \hat{\mathbf{A}})^{-1} \hat{\mathbf{A}}^{\dagger} \hat{\mathbf{V}} \hat{\mathbf{V}}^{\dagger} \hat{\mathbf{A}} (\hat{\mathbf{A}}^{\dagger} \hat{\mathbf{A}})^{-1} \quad (20.a)$$

where

$$\hat{\mathbf{V}} = [\hat{\mathbf{V}}_1 \cdots \hat{\mathbf{V}}_d] \quad (20.b)$$

$$\hat{\mathbf{A}} = [\mathbf{A}_{\hat{\theta}_1, \hat{\omega}_1} \cdots \mathbf{A}_{\hat{\theta}_d, \hat{\omega}_d}] \quad (20.c)$$

$$\hat{\mathbf{A}} = \begin{bmatrix} \hat{\lambda}_1 - \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \hat{\lambda}_d - \sigma^2 \end{bmatrix} \quad (20.d)$$

with  $\mathbf{A}_{\hat{\theta}_i, \hat{\omega}_i}$  denoting the direction-frequency vector corresponding to the estimated direction-of-arrival and center-frequency of the  $i$ 'th source.

## E. Discussion

The maximum number of sources that the method can handle is  $pm - 1$ , assuming that not more than  $m - 1$  of them are "co-frequency" (i.e., have the same center-frequency). These restrictions stem from condition (II) of our underlying assumptions which requires that any set of  $d + 1$  direction-frequency vectors should be linearly independent. Thus, since the frequency-direction vector is of dimension  $mp$ , the number of sources  $d$  must be no greater than  $pm - 1$ . The restriction on the number of co-frequency sources is because the subspace of all possible co-frequency direction-frequency vectors is of dimension no greater than the number of sensors  $m$ , regardless of the number of taps  $p$ . That is, any set of  $m$  or more co-frequency direction-frequency vectors will definitely be *linearly dependent*. Therefore, for condition (II) above to hold, the number of co-frequency sources should be no greater than  $m - 1$ .

The structure of the array for which the method is applicable is also restricted by condition (II) above. This condition does not hold in general for arbitrary geometry and arbitrary number of taps. It turns out, for example, that for the case of a uniform linear array, this condition does not hold for one tap, i.e.,  $p = 1$ . This is because all the direction-frequency vectors corresponding to points  $(\theta, \omega)$  on the curve  $\omega \sin \theta = \text{constant}$  are *identical*, and hence linearly dependent. Thus to assure the applicability of the method in the case of a uniform linear array, a tapped-delay-line with more than one tap is needed. The exact number of taps  $p$  that will guarantee the applicability of the method for a given geometry and frequency band is unknown even for the simple case of



uniform linear array. Nevertheless, the simulation results (see sec. V) show that this number is usually very small.

The method presented (with the arithmetic-mean metric) reduces to Pisarenko and Schmidt's method for the 1-D problem, that is, for the case that all the sources have the same known center-frequency and only one tap is used ( $p = 1$ ) (Pisarenko's method is a special case of Schmidt's method where only *one* of the eigenvectors corresponding to the smallest eigenvalue is used). Larimore's (1981) method and the variant of Reddy et al. (1982b) essentially amounts to on-line computation of the eigenvector corresponding to the smallest eigenvalue of the covariance matrix (a fact not pointed out in either paper). Thus, Their method can be regarded as an on-line implementation of the off-line 2-D extension to Pisarenko's method we have presented above.

#### IV. Frequency-domain Eigenstructure Method for the Wideband Problem

The solution we are going to present for the wideband problem is based on the the eigenstructure of the spectral-density matrix  $\mathbf{K}(\omega_n)$  and therefore is referred to as a frequency-domain method. It presumes that the following conditions hold:

- (I) The spectral-density matrix  $\mathbf{P}(\omega_n)$  is *nonsingular*.
- (II) Any set of  $d+1$  direction vectors are *linearly independent*.

With these assumptions it can be easily verified that the eigenvalues and eigenvectors of  $\mathbf{K}(\omega_n)$ , denoted by  $\lambda_1(\omega_n) \leq \dots \leq \lambda_m(\omega_n)$  and  $\mathbf{V}_1(\omega_n), \dots, \mathbf{V}_m(\omega_n)$ , respectively, have the following properties:

The minimal eigenvalue of  $\mathbf{K}(\omega_n)$  is  $\sigma^2(\omega_n)$  with *multiplicity*  $m - d$ , i.e.,

$$\lambda_{d+1}(\omega_n) = \dots = \lambda_m(\omega_n) = \sigma^2(\omega_n) \quad (21)$$

The eigenvectors corresponding to the minimal eigenvalue are *orthogonal* to the columns of the matrix  $\mathbf{A}(\omega_n)$ , i.e.,

$$\mathbf{A}(\omega_n)^\dagger \mathbf{V}_i(\omega_n) = 0 \quad i = d+1, \dots, m \quad (22.a)$$

or, more explicitly

$$\{\mathbf{V}_{d+1}(\omega_n), \dots, \mathbf{V}_m(\omega_n)\} \perp \{\mathbf{A}_{\theta_1}(\omega_n), \dots, \mathbf{A}_{\theta_d}(\omega_n)\} \quad (22.b)$$

where  $\mathbf{A}_{\theta_k}(\omega_n)$  is the  $k$ 'th column of the matrix  $\mathbf{A}(\omega_n)$ , referred to as the direction-vector of the  $k$ 'th source. Note that the orthogonality relation (22) holds for every  $\omega_n$  that is included in the signal bandwidth  $B$ .

These relations form the basis for the frequency-domain eigenstructure method. Because of the analogy with the time-domain method we shall proceed directly to the description of the method.

### A. Determination of the Number of Sources

As in the time-domain method, the number of sources can be determined from the multiplicity of the smallest eigenvalue of the spectral density matrix  $\mathbf{K}(\omega_n)$ . Applying the Bartlett-Lawley procedure, the hypothesis tested at the  $k$ 'th step is given by

$$H_k(\omega_n) : \lambda_{k+1}(\omega_n) = \dots = \lambda_m(\omega_n)$$

Following a well-known analogy between multivariate analysis and time-series analysis (see, e.g., Brillinger (1964) and Wahba (1968)), the likelihood-ratio for this hypothesis is obtained simply by substituting the eigenvalues of the periodogram estimate of the spectral-density matrix for those of the sample-covariance matrix, that is

$$Q_k(\omega_n) = \left[ \frac{\prod_{i=k+1}^m l_i(\omega_n)}{\left( \frac{1}{m-k} \sum_{i=k+1}^m l_i(\omega_n) \right)^{m-k}} \right]^L \quad (23)$$

where  $l_1(\omega_n) \geq l_2(\omega_n) \geq \dots \geq l_m(\omega_n)$  denote the eigenvalues of the periodogram estimate of the spectral density matrix

$$\hat{\mathbf{K}}(\omega_n) = \frac{1}{L} \sum_{i=1}^L \mathbf{R}_i(\omega_n) \mathbf{R}_i^H(\omega_n) \quad (24)$$

with  $\mathbf{R}_i(\omega_n)$  denoting the Fourier-coefficient vector of the  $i$ -th subinterval at the frequency  $\omega_n$  and  $L$  denoting the number of subintervals.

Since the sources are wideband, the number of sources can be estimated from *any* frequency  $\omega_n \in B$ . Thus, to minimize the probability of error, the following composite hypothesis, which captures the information in *all* the frequencies  $\omega_n \in B$ , should be tested:

$$H_k : \quad \lambda_{k+1}(\omega_n) = \dots = \lambda_m(\omega_n) \quad \text{for every } \omega_n \in B$$

Assuming that the observation time  $T$  is much larger than the processes correlation time, it follows (see, e.g., Whalen p.81) that the Fourier-coefficients corresponding to different frequencies are uncorrelated. Thus, under Gaussian assumptions, the composite likelihood-ratio test is given by

$$Q_k = \prod_{n=1}^{l+M} Q_k(\omega_n) \quad (25)$$

The statistic  $-2 \ln Q_k$  has in this case an asymptotic  $\chi^2$  distribution with  $M[(m-k)^2 - 1]$  degrees of freedom. Therefore, if  $N$  is large an approximate test of size  $\alpha$  of  $H_k$  is to reject  $H_k$  if  $-2 \ln Q_k > c(\alpha; (m-k)^2 - 1)$ , where  $c(\alpha; r)$  is the upper  $100\alpha$  % point of the  $\chi_r^2$  distribution.

The hypotheses  $H_k$  ( $k = 0, 1, \dots, m-1$ ) are tested sequentially. The value of  $k$  for which  $H_k$  is first accepted is selected as the estimate  $\hat{d}$  of the number of sources. For the 1-D problem ( $M=1$ ) this procedure reduces to that presented by Liggett (1972).

We should note that a different approach to the problem, based on applying the information theoretic criteria for statistical model identification introduced by

Akaike, Schwartz and Rissanen, was recently presented by Wax and Kailath (1983b). In this approach no subjective judgement is required in the decision process.

## B. Estimation of the Eigenstructure

Let  $l_1(\omega_n) \geq l_2(\omega_n) \geq \dots \geq l_m(\omega_n)$  and  $C_1(\omega_n), \dots, C_m(\omega_n)$  denote the eigenvalues and eigenvectors, respectively, of the spectral density matrix  $K(\omega_n)$ . Following again the duality between multivariate analysis and time-series analysis, it follows that the maximum likelihood estimates of the eigenvalue and eigenvectors of  $K(\omega_n)$  are given by

$$\hat{\lambda}_i(\omega_n) = l_i(\omega_n) \quad i = 1, \dots, \hat{d} \quad (26.a)$$

$$\hat{\sigma}^2(\omega_n) = \frac{1}{m - \hat{d}} \sum_{i=\hat{d}+1}^m l_i(\omega_n) \quad (26.b)$$

$$\hat{V}_i(\omega_n) = C_i(\omega_n) \quad i = 1, \dots, \hat{d} \quad (26.c)$$

and up to an orthogonal transformation from the right

$$\hat{V}_i(\omega_n) = C_i(\omega_n) \quad i = \hat{d} + 1, \dots, m \quad (26.d)$$

## C. Estimation of the Directions-Of-Arrival

By analogy with the time domain method, the  $\hat{d}$  directions-of-arrival should be chosen as those whose direction vectors are "most nearly orthogonal" to the set of eigenvectors  $\{\hat{V}_k(\omega_n), k = 1, \dots, m - \hat{d}\}$ . Note that this "distance from orthogonality" should be measured for *all* the frequency bins  $\omega_n \in B$ . Thus, we must first measure the "distance from orthogonality" at each frequency bin, and then combine the resulted measures for the different frequencies. Using the arithmetic-mean metric both for the individual frequency bins and for the

combination over the frequency range, yields the estimator (the subscript 11 stands for the double use of the arithmetic-mean metric), given by

$$J_{11}(\theta) = \frac{|\mathbf{A}^T(\omega_n)\mathbf{A}_d(\omega_n)|}{\frac{1}{M} \sum_{n=l}^{l+M} \frac{1}{m-d} \sum_{k=d+1}^m |\mathbf{A}^T(\omega_n)\hat{\mathbf{V}}_k(\omega_n)|^2} \quad (27.a)$$

Using the arithmetic-mean metric for the individual frequency bins and the geometric-mean metric for the combination over the frequency range, yields the estimator

$$J_{10}(\theta) = \frac{|\mathbf{A}^T(\omega_n)\mathbf{A}_d(\omega_n)|}{\prod_{n=l}^{l+M} \left( \frac{1}{m-d} \sum_{k=d+1}^m |\mathbf{A}^T(\omega_n)\hat{\mathbf{V}}_k(\omega_n)|^2 \right)^{\frac{1}{M}}} \quad (27.b)$$

The directions-of-arrival of the sources are then determined as the  $\hat{d}$  points  $\theta_1, \dots, \theta_{\hat{d}}$  that yield the highest peaks of either one of the estimators given in (27).

Note that other combinations of the arithmetic-mean and the geometric-mean metrics, as well as other combinations of different metrics from the general family of metrics introduced in section III can be used. From the corresponding discussion for the time-domain method it follows that the estimator (27.b) exhibits higher resolution and lower accuracy than the estimator (27.a).

#### D. Estimation of the Spectral Density Matrix of the Sources

The spectral density matrix of the sources  $\mathbf{P}(\omega_n)$  contains valuable information on the impinging sources. The diagonal elements yield the spectral

densities of the sources, and thus provides a tool for classifying the sources according to their spectral "signatures". The off-diagonal elements indicate the amount of correlation existing between the sources, and thus provide a way to distinguish between a direct path and multipath, for example.

The estimation scheme is analogous to that described for the time-domain method. Therefore by analogy with (20), we obtain

$$\hat{\mathbf{P}}(\omega_n) = [\hat{\mathbf{A}}^\dagger(\omega_n) \hat{\mathbf{A}}(\omega_n)]^{-1} \hat{\mathbf{A}}^\dagger(\omega_n) \hat{\mathbf{V}}(\omega_n) \hat{\mathbf{A}}(\omega_n) \hat{\mathbf{V}}^\dagger(\omega_n) \hat{\mathbf{A}}(\omega_n) [\hat{\mathbf{A}}^\dagger(\omega_n) \hat{\mathbf{A}}(\omega_n)]^{-1} \quad (28.a)$$

where

$$\hat{\mathbf{V}}(\omega_n) = [\hat{\mathbf{V}}_1(\omega_n) \cdots \hat{\mathbf{V}}_d(\omega_n)] \quad (28.b)$$

$$\hat{\mathbf{A}}(\omega_n) = \begin{bmatrix} \hat{\lambda}_1(\omega_n) - \hat{\sigma}^2(\omega_n) & & \\ & \ddots & \\ & & \hat{\lambda}_d(\omega_n) - \hat{\sigma}^2(\omega_n) \end{bmatrix} \quad (28.c)$$

and

$$\hat{\mathbf{A}}(\omega_n) = [\mathbf{A}_{\hat{j}_1}(\omega_n) \cdots \mathbf{A}_{\hat{j}_d}(\omega_n)] \quad (26.d)$$

with  $\mathbf{A}_{\hat{j}_i}(\omega_n)$  denoting the direction-vector corresponding to the estimated direction-of-arrival of the  $i$ -th source.

## E. Discussion

The maximum number of sources the method can handle is  $m - 1$ . From the requirement that any set of  $d + 1$  direction-vectors be linearly indepen-



dent, it follows that since the direction-vector is of dimension  $m$ , the number of sources should be no greater than  $m - 1$ .

The structure of the array for which the frequency-domain method is applicable is restricted by condition that any  $d + 1$  direction-vectors be linearly independent. Note, however, that for a omnidirectional ( $a_{ik} = 1$ ) uniform linear array, this condition is always satisfied, because of the Vandermonde type of the direction-vectors.

The estimation of the spectral-density matrix of the received signals is by no means restricted to the peridogram method. Any multivariate spectral estimation technique, parametric or nonparametric can be used if there is reason to believe that it will give better results.

The frequency-domain eigenstructure method we have presented reduces to Bienvenu and Kopp's method for the 1-D problem, that is, when the sources occupy only *one* frequency bin ( $M=1$ ).

## V. Simulations Results

In this section we present computer simulation results that illustrate the performance of the proposed methods. All the examples will refer to a uniform linear array. For this type of array it is convenient to use the notion of normalized frequency, defined as  $\frac{\omega \sin \theta \Delta}{c}$ , where  $\omega$  is the frequency of the source,  $\theta$  is its direction-of-arrival,  $\Delta$  is the spacing between the sensors, and  $c$  is the speed of propagation.

In the first example we wanted to demonstrate the time-domain method for the narrowband problem. We considered 2 sinusoidal sources, having normalized frequencies 0.2 and 0.3 and normalized wavenumbers 0.125 and 0.2, respectively, impinging on a uniform linear array of 3 sensors ( $m=3$ ), each followed by a tapped-delay-line with 3 taps ( $p=3$ ). The signal-to-noise ratio was 10dB. The results, obtained from 200 "snapshots", using the estimator (18.a), are presented in Fig. 2. The two peaks corresponding to the two sources are clearly seen.

In the second example we wanted to demonstrate the ability of the time-domain method to resolve more sources than the number of sensors, given that the number of taps is appropriate. We considered 4 sinusoidal sources, having normalized frequencies 0.1, 0.2, 0.3 and 0.4 and normalized wavenumbers 0.125, 0.25, 0.35, and 0.42, respectively, impinging on a uniform linear array of 3 sensors ( $m=3$ ), each followed by a tapped-delay-line of 3 taps ( $p=3$ ). The signal-to-noise ratio was 10dB. The results, obtained from 1000 "snapshots",

using the estimator (18.a), are presented in Fig.3. The four peaks corresponding to the four sources are clearly seen.

In the third example we wanted to demonstrate the frequency-domain method for the wideband problem. We considered 3 wideband sources having identical spectra, centered at 0.25 with bandwidth 0.05, impinging from  $36^\circ$ ,  $60^\circ$ , and  $120^\circ$  on a uniform linear array of 5 sensors ( $m=5$ ) spaced a third wavelength apart. The signal-to-noise ratio was 6dB. The results obtained from 100x64 samples, using the estimator (27.a) are presented in Fig. 4 (a 64 point DFT was used to compute the spectral-density matrix). The three peaks corresponding to the three sources are clearly seen.

## VI. Concluding Remarks

The eigenstructure methods developed by Pisarenko (1973), Schmidt (1979) and Bienvenu and Kopp (1980), were confined to the 1-D problem. In this paper we have generalized these methods to the 2-D problem, that is, to the simultaneous estimation of the spatio-temporal spectrum of the signals received by the passive array. We have presented both a time-domain method, based on the eigenstructure of the covariance matrix of the received signals, and a frequency-domain method, based on the eigenstructure of the spectral-density matrix of the received signals. Though the time-domain method was applied to the narrowband problem and the frequency-domain method was applied to the wideband problem, the methods are in fact applicable to both problems. The applicability of the frequency-domain method to the narrowband case can be seen by noting that the wideband problem includes the narrowband problem as a special case corresponding to  $M = 1$ . The applicability of the time-domain method to the wideband problem can be seen by noting that from the Fourier-representation it follows that a wideband signal can be decomposed to a weighted sum of  $M$  complex exponentials. Thus, the applicability of the time-domain method to the wideband problem is guaranteed if the number of taps  $p$  is at least equal to  $M$ . However, the computational complexity involved in implementing the time-domain method for the wideband problem and the frequency-domain method for the narrowband problem, make these alternatives unattractive in practice.

The resolution offered by the eigenstructure methods depends crucially on the quality of the estimates of the covariance and spectral-density matrices. If the

data record is long enough to enable reasonably good estimates of these matrices, the resolution is very high even in relatively low signal-to-noise ratios. However, if the data record is too short, the performance of these methods deteriorate drastically.

Theoretical analysis of the performance of these method for finite record length seems to be a very difficult problem. The only analytical results available are for the asymptotic where the record length approaches infinity. It has been recently shown by Sharman et al (1983) that the eigenstructure estimator based on the arithmetic-mean metric is asymptotically unbiased, and that for the case of a single source, it is even efficient, that is, its variance approaches the Cramer-Rao lower bound.

The methods we have presented can be straightforwardly extended to the case that the sources have diverse polarization (see Schmidt (1979) and Ferrara and Parks (1983)) and to the more general problem of source localization (see Wax et. al (1982)).

#### Acknowledgement

The authors wish to acknowledge helpful discussions with Dr. B. Medoff.

## REFERENCES

- [1] Anderson, T. W. (1963): "Asymptotic Theory for Principal Component Analysis," *Ann. of Math. Stat.*, Vol. 34, No. 1, 1963, pp. 122-148.
- [2] Bienvenu, G. and Kopp, L. (1980): "Adaptivity to Background Noise Spatial Coherence for High Resolution Passive Methods", in *Proc. IEEE ICASSP 80*, (Denver, CO), pp. 307-310.
- [3] Bienvenu, G. and Kopp, L. (1981): "Source Power Estimation Method Associated with High Resolution Bearing Estimation", in *Proc. IEEE ICASSP*, (Atlanta, GA), pp. 302-305.
- [4] Brillinger, D.R. (1964): "A Frequency Approach to the Techniques of Principle Components, Factor Analysis and Canonical Variates in the Case of Stationary Series," *Royal Statistical Society Conference*, (Cardiff, Wales).
- [5] Bronez, T.P. and Cadzow, J.A. (1983): "An Algebraic Approach to super-resolution Array Processing," *IEEE Trans. on AES*, Vol 19, No 1, pp.123-133.
- [6] Cantoni, A. and Godara, L.C. (1980): "Resolving the Directions of the Sources in a Correlated Field Incidenting on an Array," *JASA*, Vol. 67, No. 4, pp. 1247-1255.
- [7] Cox, D.R., and Hinkley, D.V. (1974): *Theoretical Statistics*. Chapman and Hall, London.
- [8] Ferrara, E.R and Parks, T.M. (1983): "Direction Finding with an Array of Antennas Having Diverse Polarizations," *IEEE Trans. on AP*, Vol 31, No 2, pp.231-236.
- [9] Frost, O.L. and Sullivan, T.M. (1979): "High Resolution Two-Dimensional Spectral Analysis," in *Proc. ICASSP 79*, (Washington, DC), pp. 673-676.
- [10] Hahn, W.R. and Tretter, S.A. (1973): "Optimum Processing for Delay-Vector Estimation in Passive Signal Array," *IEEE Trans. on IT*, Vol. 19, No. 5, pp. 608-614.
- [11] Jackson, L.B. and Chien, C.H. (1979): "Frequency and Bearing Estimation by Two-Dimensional Linear Prediction," in *Proc. IEEE ICASSP 79*, (Washington, DC), pp. 665-668.
- [12] Jain, A.K. and Raganath, S. (1978): "Two-Dimensional Spectral Estimation," in *Proc. RDAC Spectrum Estimation Workshop*, (Rome, N.Y), pp. 151-157.

- [13] Johnson, D.H. and Degraff, S.R. (1982): "Improving the Resolution of Bearing in Passive Sonar Arrays by Eigenvalue Analysis," *IEEE Trans. on ASSP*, Vol 30, No 4, pp. 638-647.
- [14] Kumaresan, R. and Tufts, D.W. (1981): "A Two-Dimensional Technique for Frequency-Wavenumber Estimation," *Proc. IEEE (Lett.)*, Vol 69, No 11, pp. 1515-1517.
- [15] Kumaresan, R. and Tufts, D.W. (1983): "Estimation of Arrival of Multiple Plane Waves," *IEEE Trans. on AES*, Vol 19, No 1, pp. 134-139.
- [16] Larimore, M.G. (1981): "Adaptive Two-dimensional Harmonic Retrieval," *Proc. 15th Asilomar Conf. Cir. Syst. Comp.*, November 1981.
- [17] Ligget, W.S. (1972): "Passive Sonar: Fitting Models to Multiple Time Series," in *Nato ASI on Signal Processing*, (Loughborough, U.K.), pp. 327-345.
- [18] Lim, J.S. and Malik, N.L. (1981): "A New Algorithm for Two-Dimensional Maximum Entropy Power Spectrum Estimation," *IEEE Trans. on ASSP*, Vol 29, No 3, pp. 401-413.
- [19] Lawley, D.N., (1956): "Tests of Significance of the Latent Roots of the Covariance and Correlation Matrices," *Biometrika*, 43, pp. 128-136.
- [20] Good, I.J. (1963): "Weighted Covariance for Estimating the Direction of a Gaussian Source," in *Time Series Analysis*, M. Rosenblat Ed., New York, Wiley, 1963.
- [21] McClellan, J.H. (1982): "Multidimensional Spectral Estimation," *Proc. IEEE*, Vol 70, No 9, pp. 1029-1039.
- [22] Morf, M. et al. (1981): "Investigation of New Algorithms for Locating and Identifying Spatially Distributed Sources and Receivers," DARPA Tech. Rep. No. M355-1.
- [23] Nehorai, A. and Morf, M. (1982): "Estimation of Time Difference of Arrival for Multiple ARMA Sources by Pole Decomposition," *IEEE Trans on ASSP*, Vol 31, No 6.
- [24] Owsley, N.L. (1973): "A Recent Trend in Adaptive Spatial Processing for Sensors Arrays: Constrained Adaptation," in *Signal Processing*, (J.W.R. Griffiths et al. Eds.), New York: Academic Press.
- [25] Pisarenko, V.F. (1972): "On the Estimation of Spectra by Means of Non-linear Functions of the Covariance Matrix," *Geophys. J. R. astr. Soc.*, Vol 28, pp. 511-531.

- [26] Pisarenko, V.F. (1973): "The Retrieval of Harmonics from Covariance Functions," *Geoph. J. Royal Astronomical Soc.*, 1972, pp. 347-366
- [27] Porat, B. and Friedlander, B. (1983): "Estimation of Spatial and Spectral Parameters of Multiple Sources," *IEEE Trans on IT*, Vol 29, No 3, pp. 412-425.
- [28] Reddi, S.S. (1979): "Multiple Source Location - A digital Approach," *IEEE Trans. on AES*, Vol 15, No 1, pp. 95-105.
- [29] Reddy, V.U., Egardt, B. and Kailath, T. (1982): "Least-squares Type Algorithm for Adaptive Implementation of Pisarenko's Harmonic Retrieval Method," *IEEE Trans. on ASSP*, Vol. 30, No. 3, June 1982, pp. 399-405.
- [30] Reddy, V.U., Citron, T.K and Kailath, T. (1982): "Adaptive Algorithms for Two-Dimensional Harmonic Retrieval," in *Proc. 16th Asilomar Conf. Cir., Syst. Comp.*, (Pacific Groove, CA), pp. 177-181.
- [31] Roucos, S. and Childers, D.G. (1980): "A Two-Dimensional Maximum Entropy Spectral Estimator," *IEEE Trans. on IT*, Vol 26, No 5, pp. 554-560.
- [32] Sharman, K., Wax, M. and Kailath, T. (1983): "Asymptotic Statistics of the Eigenstructure Spectral Analysis Method," submitted to *IEEE Trans. on ASSP*
- [33] Schmidt, R.O. (1979): "Multiple Emitter Location and Signal Parameter Estimation," in *Proc. RADC Spectrum Estimation Workshop*, (Rome, N.Y), pp. 243-258.
- [34] Schmidt, R.O. (1981): "A signal subspace approach to multiple emitter location and spectral estimation," Ph.D. dissertation, Stanford University, November 1981.
- [35] Schweppe, F.C. (1968): "Sensor Array Data Processing for Multiple Signal Sources," *IEEE Trans. on IT*, Vol 14, pp. 294-305.
- [36] Simkins, D.N. (1980): "Multichannel Angle-Of-Arrival Estimation," *Ph.D Dissertation*, Stanford University.
- [37] Thompson, P.A. (1979): "An Adaptive Spectral Analysis Technique for Unbiased Frequency Estimation in the Presence of White Noise," in *Proc. 13th Asilomar Conf. Cir., Syst. Comp.*, (Pacific Groove, CA), pp. 529-533
- [38] Wahba, G., (1968): "On the Distribution of Some Statistics Useful in the Analysis of Jointly Stationary Time-Series," *Ann. Math. Stat.*, Vol 39, pp. 1849-1862.



- [39] Wax, M., Shan, T-J. and Kailath, T. (1982): "Location and Spectral Density Estimation of Multiple Sources," *Proc. 16th Asilomar Conf. Cir., Syst. Comp.*, (Pacific Grove, CA), pp. 322-326.
- [40] Wax, M. and Kailath, T. (1983a): "Optimum Localization of Multiple Sources by Passive Arrays," *IEEE Trans. on ASSP*. Vol 31, No 5, pp. 1210-1218.
- [41] Wax, M. and Kailath, T. (1983b): "Determining the Number of Signals by Information Theoretic Criteria," *ASSP Spectrum Estimation Workshop II*, (tampa, FL), pp. 192-196.
- [42] Whalen, A. D. (1971): *Detection of Signals in Noise*, Academic Press, New York.

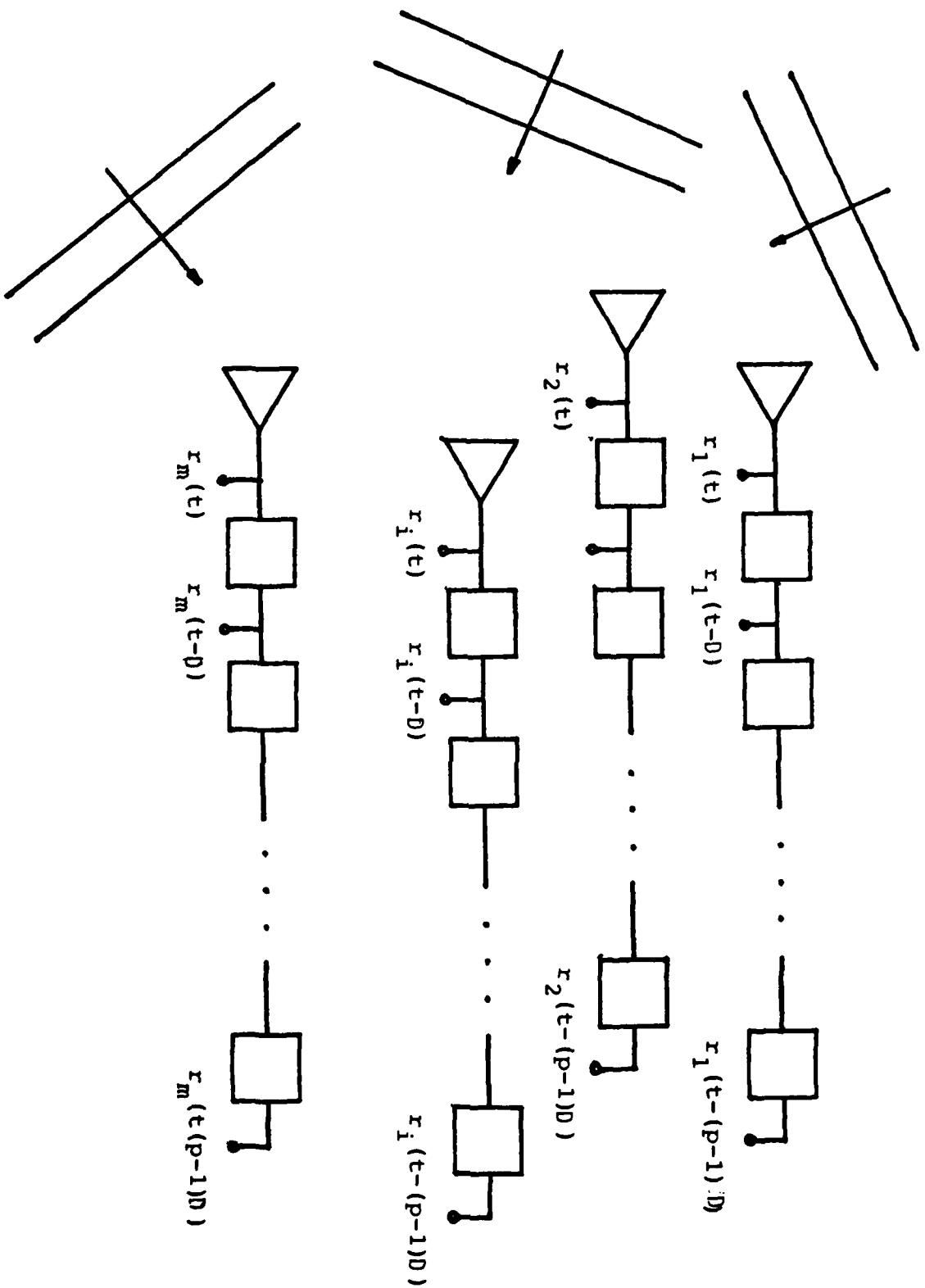


Figure 1: The Array Configuration

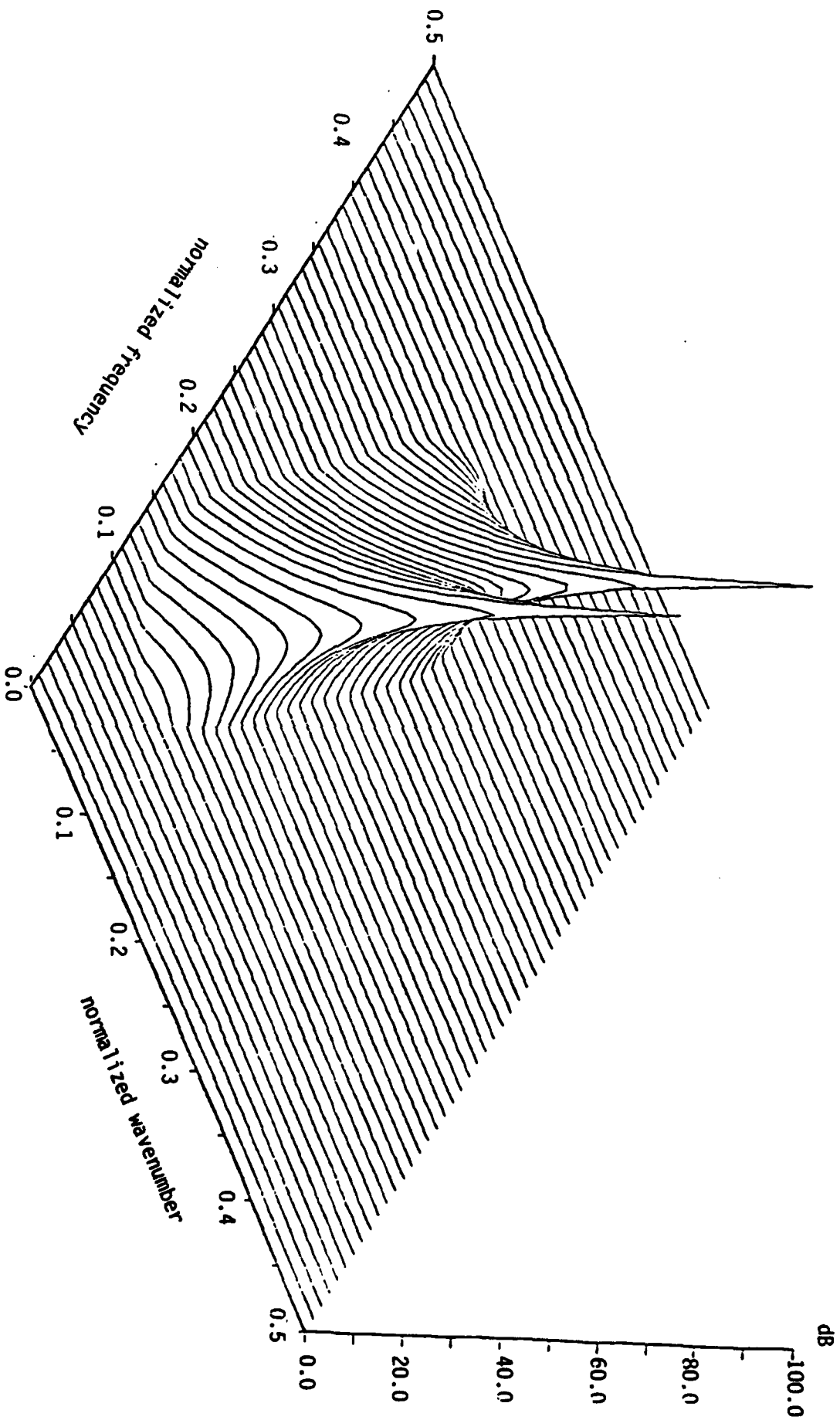


Figure 2. 2 sinusoidal sources having normalized frequencies 0.2 and 0.3 and wavenumbers 0.125 and 0.2, respectively, impinge on a uniform linear array of 3 sensors ( $m=3$ ) followed by a tapped-delay line with 3 taps ( $P=3$ ). The signal-to-noise ratio was 10dB. The results were obtained from 200 "snapshots", using the estimator (17.a).

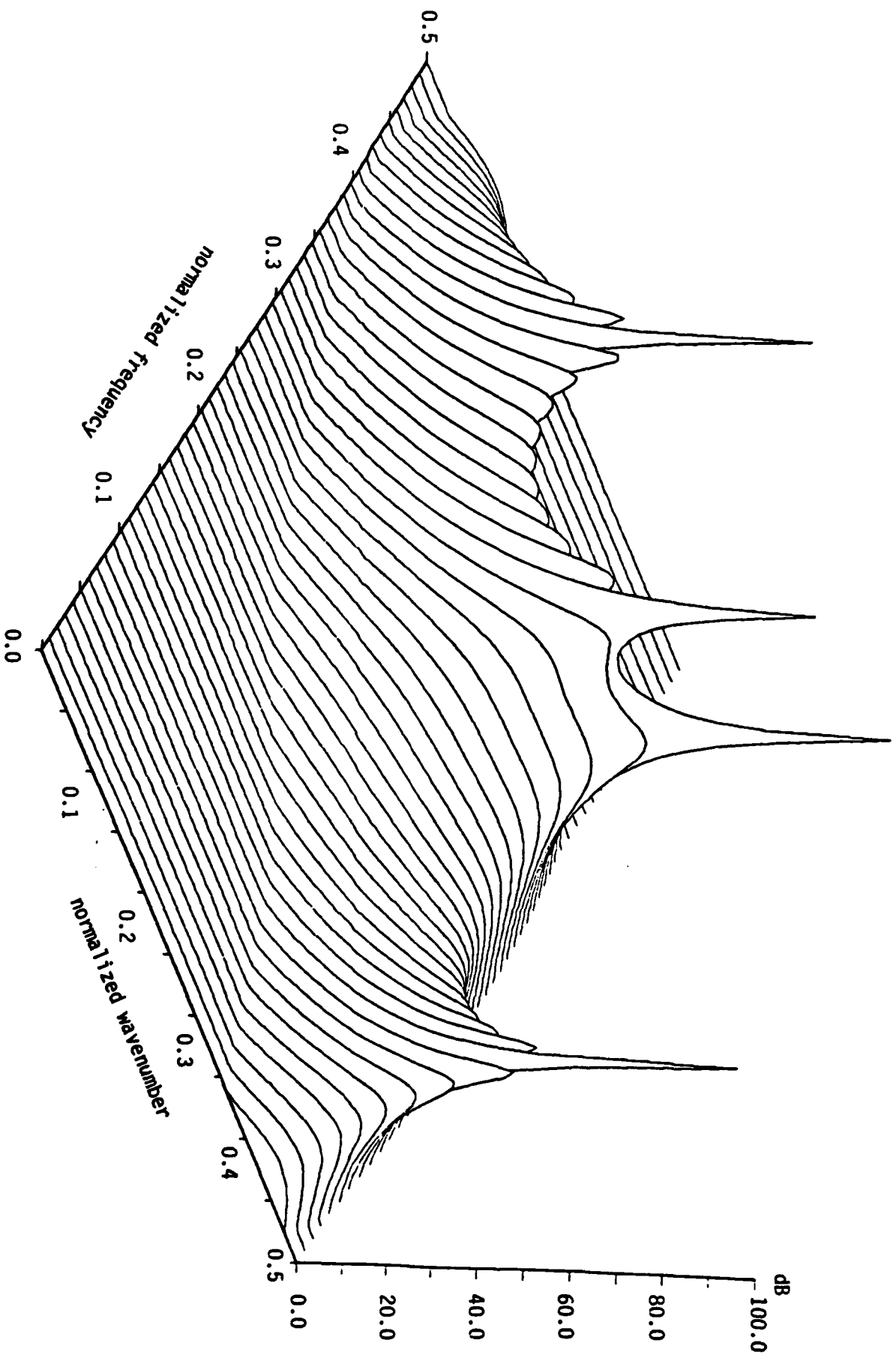


Figure 3. 4 sinusoidal sources having normalized frequencies 0.1, 0.2, 0.3 and 0.4 and normalized wavenumbers 0.125, 0.25, 0.35 and 0.42, respectively, impinge on uniform linear array of 3 sensors ( $m=3$ ) followed by a tapped-delay-line of 3 taps ( $p=3$ ). The signal-to-noise ratio was 10dB. The results were obtained from 1000 "snapshots", using estimation (17.a).

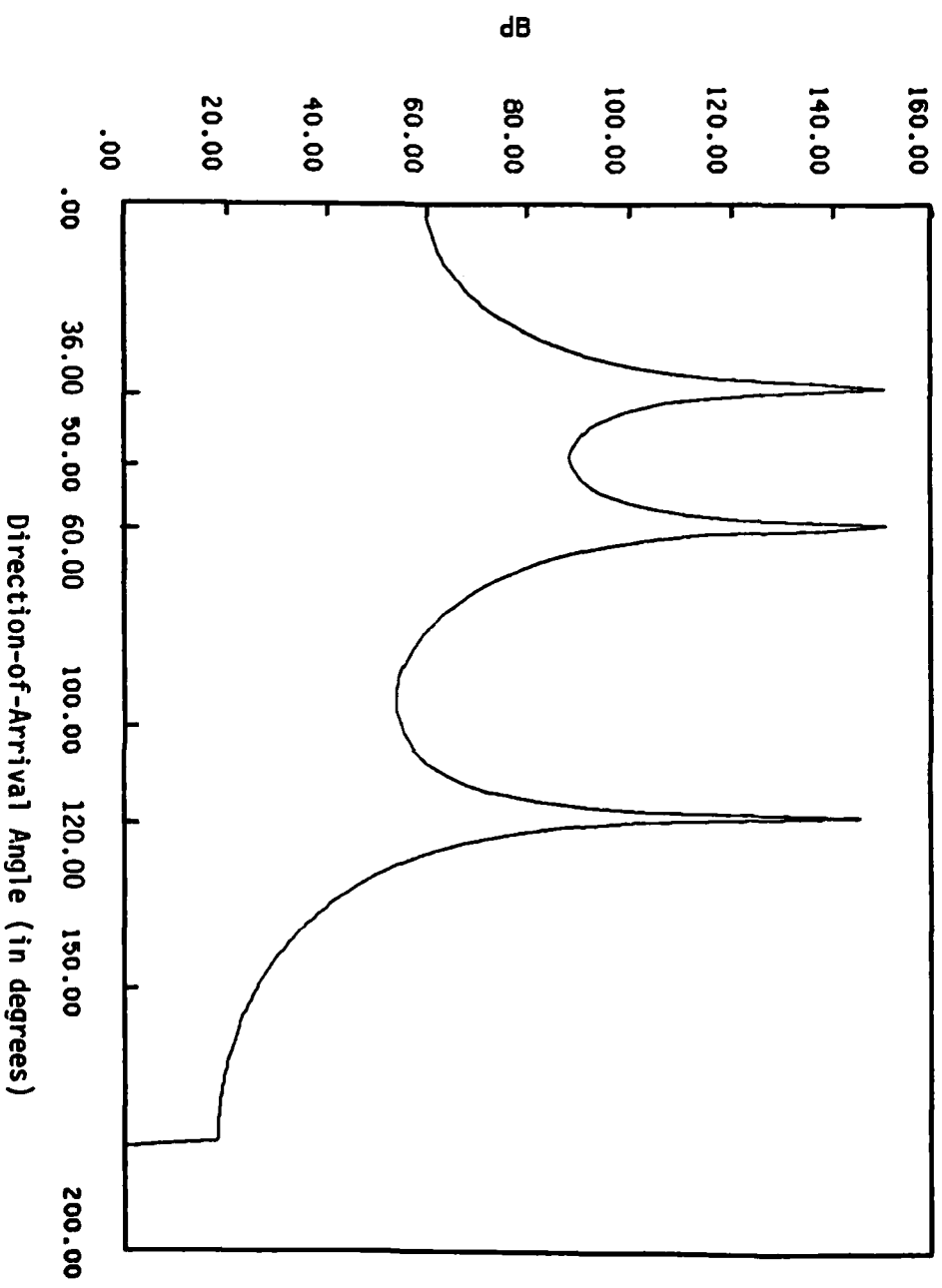


Figure 4. 3 wideband sources having identical spectra centered at 0.25 with bandwidth 0.05, impinge from 36°, 60°, and 120° on a uniform linear array of 5 sensors ( $m=5$ ) spaced a third wavelength apart. The signal-to-noise ratio was 6dB. The results were obtained from 64x100 samples, using estimator (25.a). (A 64 DFT was used to compute the estimate of the spectral-density matrix.)

END

FILMED

9-84

DTIC